The parallel refinement of weakening idempotent pair

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Abstract

In this paper, we introduce the parallel refinement of weakening idempotent pair and conduct a quantitative matrix analysis for this refinement. Our analysis shows that this refinement is more effective than the known one when one applies weakening idempotent pairs to K-theory.

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1. Introduction

Recently, people are interested in weakening idempotent pair, especially its application in K-theory [1, 2]. Given self-adjoint $A_+, A_- \in M_n(\mathbb{C})$ (or in the more general setting of C^*-algebra), we call (A_+, A_-) an (ε-)*weakening idempotent pair* if it holds that

$$\|(A_\pm - A_\pm^2)(A_+ - A_-)\| < \varepsilon, \tag{1}$$

for some small $\varepsilon > 0$. However, it doesn't necessarily hold that

$$0 \leq A_+, A_- \leq 1. \tag{2}$$

If (2) holds, by the formula

$$Q = \begin{pmatrix} 1 - A_+ & \kappa(A_+) \\ \kappa(A_+) & A_- \end{pmatrix} \quad (\text{where } \kappa(t) = \sqrt{t - t^2}), \tag{3}$$

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we obtain an almost projection Q that can be applied to investigate the K-theory of C^*-algebra [1] or topological space [2]. However, (2) usually fails to hold, hence we need to modify (A_+, A_-) so as to get a new pair (B_+, B_-) that both satisfies $0 \leq B_\pm \leq 1$ and preserves the K-theory information in (A_+, A_-). We call this process the *refinement* of weakening idempotent pair.

It is obvious that such refinement is not unique. For example, $A_\pm \in C(X, M_{mn}(\mathbb{C}))$ are defined in [2] from generalized pairs of cocycles $\{g^\pm_{\alpha\beta}\}_{\alpha,\beta\in I}$, which satisfy that

$$\|(A_+ - A_+^2)(A_+ - A_-)\| < m\varepsilon, \quad \|(A_- - A_-^2)(A_+ - A_-)\| < m\varepsilon \tag{4}$$

and

$$\|A_+\| \leq m, \quad \|A_-\| \leq m \tag{5}$$

where $m = |I|$. So, (A_+, A_-) is an ($m\varepsilon$-)weakening idempotent pair.

To refine $A_\pm$, let f be the function on $\mathbb{R}$ given by

$$f(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ t, & \text{if } 0 \leq t \leq 1; \\ 1, & \text{if } t \geq 1. \end{cases},$$

and set $B_\pm = f(A_\pm)$, then it is trivial that $0 \leq B_\pm \leq 1$. It is proved in [2] that

$$\|(B_\pm - B_\pm^2)(B_+ - B_-)\| < 2C(m)\sqrt[4]{\varepsilon}, \tag{6}$$

where $C(m)$ is a function of the form "$2m \ln 16m\sqrt{2(m+3)}\sqrt[4]{m}$". From the functional calculus of f over $A_\pm$ and the estimation (6), we can see that $B_\pm$ coordinate with each other to remain the "idempotent-like" part of $A_\pm$, hence they preserve the K-theory information in $A_\pm$.

The function $C(m)$ in the estimation inequality (6) is rather big and complicated, partly because $B_\pm$ are achieved separately by the functional calculus of f so that $B_\pm$ won't coordinate with each other in $(B_\pm - B_\pm^2)(B_+ - B_-)$ as well as $A_\pm$ originally do in $(A_\pm - A_\pm^2)(A_+ - A_-)$. Therefore, instead of applying the functional calculus of f to A_+ and A_- each directly, this paper invents a new refinement procedure, which step by step removes the non-idempotent part from A_+ and A_- simultaneously and finally obtain $0 \leq C_\pm \leq 1$ such that

$$\|(C_\pm - C_\pm^2)(C_+ - C_-)\| < 6m^{2/3}\varepsilon^{1/3}. \quad (7)$$

Obviously, the estimation (7) is much better than (6), both in the form of $C(m)$ and the exponential degree of ε. Since A_+ and A_- are handled simultaneously, we call this method the *parallel refinement* of weakening idempotent pair. Quantitative matrix analysis details of this method are demonstrated in the following section.

2. The parallel refinement of weakening idempotent pair (A_+, A_-)

Suppose that there are self-adjoint matrices $A_\pm \in M_n(\mathbb{C})$ such that

$$\|(A_\pm - A_\pm^2)(A_+ - A_-)\| < \varepsilon$$

and

$$\|A_+\| \leq m, \quad \|A_-\| \leq m.$$

Step 1. Assume that $\lambda_- < 0$ and $\lambda_+ > 1$ are solutions to the equation

$$\lambda - \lambda^2 = -\sqrt{\varepsilon}.$$

Actually, when ε is sufficiently small, we have $\lambda_- \approx -\sqrt{\varepsilon}$ and $\lambda_+ \approx 1+\sqrt{\varepsilon}$. Since A_+ is self-adjoint, we can diagonalize it and suppose that it has the following form

$$A_+ = \begin{pmatrix} A_{+11} & 0 \\ 0 & A_{+22} \end{pmatrix},$$

where A_{+11} consists of eigenvalues those are either smaller than λ_- or bigger than λ_+, while A_{+22} consists of other eigenvalues. Consequently, we have

$$M = A_+ - A_+^2 = \begin{pmatrix} M_{11} & 0 \\ 0 & M_{22} \end{pmatrix},$$

where $\|M_{11}\| > \sqrt{\varepsilon}$. Under the same base, set

$$N = A_+ - A_- = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}.$$

From the fact that $\|MN\| < \varepsilon$ and $\|M_{11}\| > \sqrt{\varepsilon}$, it implies that

$$\|N_{11}\|, \|N_{12}\|, \|N_{21}\| < \sqrt{\varepsilon}, \tag{8}$$

which means that A_{+11}, A_{+12}, A_{+21} are close to A_{-11}, A_{-12}, A_{-21}, respectively. By setting the diagonal entries of A_{+11} those who are smaller than λ_- to be 0 and those who are bigger than λ_+ to be 1, we obtain A'_+. Because of (8), we can set

$$N' = \begin{pmatrix} 0 & 0 \\ 0 & N'_{22} \end{pmatrix},$$

where $N'_{22} = N_{22}$. Then we define $A'_- = A'_+ - N'$, so we have

$$\|(A'_+ - A'^2_+)(A'_+ - A'_-)\| = \left\| \begin{pmatrix} 0 & 0 \\ 0 & M_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & N_{22} \end{pmatrix} \right\| = \|M_{22}N_{22}\| < \varepsilon. \tag{9}$$

Next, we want to estimate $\|(A'_- - A'^2_-)(A'_+ - A'_-)\|$.

Lemma 1. *$\|(A'_- - A'^2_-)(A'_+ - A'_-)\| < (2 + 4m)\varepsilon$.*

Proof. Since

$$\begin{aligned}(A'_- - A'^2_-)(A'_+ - A'_-) &= \begin{pmatrix} A'_{-11} - A'^{\,2}_{-11} & 0 \\ 0 & A'_{-22} - A'^{\,2}_{-22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & N'_{22} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & (A'_{-22} - A'^{\,2}_{-22})N'_{22} \end{pmatrix},\end{aligned}$$

we have $\|(A'_- - A'^2_-)(A'_+ - A'_-)\| = \|(A'_{-22} - A'^{\,2}_{-22})N'_{22}\| = \|(A_{-22} - A^2_{-22})N_{22}\|$. So we need to estimate $\|(A_{-22} - A^2_{-22})N_{22}\|$. Since

$$\begin{aligned}&(A_- - A_-{}^2)(A_+ - A_-) \\ &= \begin{pmatrix} A_{-11} - A^2_{-11} - A_{-12}A_{-21} & A_{-12} - A_{-11}A_{-12} - A_{-12}A_{-22} \\ A_{-21} - A_{-21}A_{-11} - A_{-22}A_{-21} & A_{-22} - A^2_{-22} - A_{-21}A_{-12} \end{pmatrix} \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix},\end{aligned}$$

we have

$$\|(A_{-21} - A_{-21}A_{-11} - A_{-22}A_{-21})N_{12} + (A_{-22} - A_{-22}^{2} - A_{-21}A_{-12})N_{22}\| < \varepsilon.$$

Hence

$$\begin{aligned}
&\|(A_{-22} - A_{-22}^{2})N_{22}\| \\
\le&\|(A_{-21} - A_{-21}A_{-11} - A_{-22}A_{-21})N_{12} + (A_{-22} - A_{-22}^{2} - A_{-21}A_{-12})N_{22}\|+ \\
&\|(A_{-21} - A_{-21}A_{-11} - A_{-22}A_{-21})N_{12}\| + \|A_{-21}A_{-12}N_{22}\|.
\end{aligned}$$

We know that $\|A_+\|, \|A_-\| \le m$, so

$$\begin{aligned}
&\|(A_{-21} - A_{-21}A_{-11} - A_{-22}A_{-21})N_{12}\| \\
\le&(\|A_{-21}\| + \|A_{-21}\|\|A_{-11}\| + \|A_{-22}\|\|A_{-21}\|)\|N_{12}\| \\
<&(\sqrt{\varepsilon} + \sqrt{\varepsilon}m + m\sqrt{\varepsilon})\sqrt{\varepsilon} \\
=&(1+2m)\varepsilon
\end{aligned}$$

and

$$\|A_{-21}A_{-12}N_{22}\| \le \|A_{-21}\|\|A_{-12}\|\|N_{22}\| \le \sqrt{\varepsilon}\sqrt{\varepsilon}2m = 2m\varepsilon.$$

We conclude that

$$\|(A_{-22} - A_{-22}^{2})N_{22}\| < \varepsilon + (1+2m)\varepsilon + 2m\varepsilon = (2+4m)\varepsilon.$$

□

Step 2. Assume that

$$A_{+22}' = \begin{pmatrix} A_{+22}'^{(11)} & 0 \\ 0 & A_{+22}'^{(22)} \end{pmatrix},$$

where $A_{+22}'^{(11)}$ consists of diagonal entries those lie in the closed $\sqrt{\varepsilon}$-ball neighborhood of 0 and 1. We know that, when ε is sufficiently small, $\lambda_- \approx -\sqrt{\varepsilon}$ and $\lambda_+ \approx 1 + \sqrt{\varepsilon}$, thus we have

$$\|A_{+22}'^{(11)} - (A_{+22}'^{(11)})^2\| \le \sqrt{\varepsilon} \quad \text{and} \quad \sqrt{\varepsilon} < \|A_{+22}'^{(22)} - (A_{+22}'^{(22)})^2\| \le 1/4.$$

Assume that

$$M' = A'_+ - {A'_+}^2 = \begin{pmatrix} 0 & 0 \\ 0 & M'_{22} \end{pmatrix},$$

where

$$M'_{22} = \begin{pmatrix} {A'_{+22}}^{(11)} - ({A'_{+22}}^{(11)})^2 & 0 \\ 0 & {A'_{+22}}^{(22)} - ({A'_{+22}}^{(22)})^2 \end{pmatrix},$$

and that

$$N'_{22} = \begin{pmatrix} N'^{(11)}_{22} & N'^{(12)}_{22} \\ N'^{(21)}_{22} & N'^{(22)}_{22} \end{pmatrix}.$$

It follows from $\|M'_{22}N'_{22}\| < \varepsilon$ that $\|N'^{(12)}_{22}\|, \|N'^{(21)}_{22}\|, \|N'^{(22)}_{22}\| < \sqrt{\varepsilon}$. Suppose that

$$A'_{-22} = \begin{pmatrix} {A'_{-22}}^{(11)} & {A'_{-22}}^{(12)} \\ {A'_{-22}}^{(21)} & {A'_{-22}}^{(22)} \end{pmatrix},$$

we now need to estimate the spectrum of ${A'_{-22}}^{(11)}$.

Lemma 2. $\mathrm{Spec}({A'_{-22}}^{(11)}) \subseteq [-3m^{1/3}\varepsilon^{1/3}, 3m^{1/3}\varepsilon^{1/3}] \cup [1-3m^{1/3}\varepsilon^{1/3}, 1+3m^{1/3}\varepsilon^{1/3}]$.

Proof. Since $\|N'_{22}(A'_{-22} - {A'_{-22}}^2)\| < (2+4m)\varepsilon$, i.e.,

$$\left\| \begin{pmatrix} N'^{(11)}_{22} & N'^{(12)}_{22} \\ N'^{(21)}_{22} & N'^{(22)}_{22} \end{pmatrix} \begin{pmatrix} {A'_{-22}}^{(11)} - ({A'_{-22}}^{(11)})^2 - {A'_{-22}}^{(12)}{A'_{-22}}^{(21)} & {A'_{-22}}^{(12)} - {A'_{-22}}^{(11)}{A'_{-22}}^{(12)} - {A'_{-22}}^{(12)}{A'_{-22}}^{(22)} \\ {A'_{-22}}^{(21)} - {A'_{-22}}^{(21)}{A'_{-22}}^{(11)} - {A'_{-22}}^{(22)}{A'_{-22}}^{(21)} & {A'_{-22}}^{(22)} - ({A'_{-22}}^{(22)})^2 - {A'_{-22}}^{(21)}{A'_{-22}}^{(12)} \end{pmatrix} \right\|$$
$$<(2+4m)\varepsilon,$$

we have

$$\|N'^{(11)}_{22}({A'_{-22}}^{(11)} - ({A'_{-22}}^{(11)})^2 - {A'_{-22}}^{(12)}{A'_{-22}}^{(21)}) + N'^{(12)}_{22}({A'_{-22}}^{(21)} - {A'_{-22}}^{(21)}{A'_{-22}}^{(11)} - {A'_{-22}}^{(22)}{A'_{-22}}^{(21)})\|$$
$$<(2+4m)\varepsilon.$$

Hence

$$\begin{aligned} &\|N'^{(11)}_{22}({A'_{-22}}^{(11)} - {A'_{-22}}^{(11)})^2)\| \\ <&(2+4m)\varepsilon + \|N'^{(11)}_{22}{A'_{-22}}^{(12)}{A'_{-22}}^{(21)}\| + \|N'^{(12)}_{22}({A'_{-22}}^{(21)} - {A'_{-22}}^{(21)}{A'_{-22}}^{(11)} - {A'_{-22}}^{(22)}{A'_{-22}}^{(21)})\| \\ <&(2+4m)\varepsilon + 2m\varepsilon + (1+2m)\varepsilon = (3+8m)\varepsilon. \end{aligned}$$

Assume that there is certain point $\mu \in \mathrm{Spec}(A'^{(11)}_{-22})$ that lies out of the closed $3m^{1/3}\varepsilon^{1/3}$-ball neighborhood of 0 and 1, and that e is a unit eigenvector of $A'^{(11)}_{-22}$ with respect to μ. Then we have $|\mu(1-\mu)| > 9m^{2/3}\varepsilon^{2/3}$ and

$$|\mu|^2\cos^2\varphi + |1-\mu|^2\sin^2\varphi > 9m^{2/3}\varepsilon^{2/3}\cos^2\varphi + 9m^{2/3}\varepsilon^{2/3}\sin^2\varphi = 9m^{2/3}\varepsilon^{2/3}$$

for any $\varphi \in [0, 2\pi]$. By $\|N'^{(11)}_{22}(A'^{(11)}_{-22} - (A'^{(11)}_{-22})^2)\| < (3+8m)\varepsilon$, we have

$$\|(A'^{(11)}_{+22} - A'^{(11)}_{-22})(A'^{(11)}_{-22} - (A'^{(11)}_{-22})^2)e\| < (3+8m)\varepsilon,$$

i.e., $\|(A'^{(11)}_{+22}e - \mu e)\||\mu(1-\mu)| < (3+8m)\varepsilon$. Hence,

$$\|(A'^{(11)}_{+22}e - \mu e)\| < \frac{(3+8m)}{9m^{2/3}}\varepsilon^{1/3}.$$

Since the eigenvalues of $A'^{(11)}_{+22}$ lie in the closed $\sqrt{\varepsilon}$-ball neighborhood of 0 and 1, we can decompose e as $e = \xi + \eta$, where $\xi \perp \eta$, such that $\|A'^{(11)}_{+22}\xi\| < \sqrt{\varepsilon}\|\xi\|$ and $\|A'^{(11)}_{+22}\eta - \eta\| < \sqrt{\varepsilon}\|\eta\|$. Therefore,

$$\begin{aligned}
&\|-\mu\xi + (1-\mu)\eta\| \\
=&\|A'^{(11)}_{+22}\xi - \mu\xi - A'^{(11)}_{+22}\xi + A'^{(11)}_{+22}\eta - \mu\eta + \eta - A'^{(11)}_{+22}\eta\| \\
\le&\|(A'^{(11)}_{+22}e - \mu e)\| + \|A'^{(11)}_{+22}\xi\| + \|\eta - A'^{(11)}_{+22}\eta\| \\
<&\frac{(3+8m)}{9m^{2/3}}\varepsilon^{1/3} + 2\varepsilon^{1/2}.
\end{aligned}$$

From $3m^{1/3}\varepsilon^{1/3} < \|-\mu\xi + (1-\mu)\eta\| < \frac{(3+8m)}{9m^{2/3}}\varepsilon^{1/3} + 2\varepsilon^{1/2}$, we get

$$1 < \frac{(3+8m)}{27m} + \frac{2}{3m^{1/3}}\varepsilon^{1/6}, \quad (m \ge 1)$$

which is a contradiction when ε is sufficiently small. So we conclude that the spectrum of $A'^{(11)}_{-22}$ lies in the closed $3m^{1/3}\varepsilon^{1/3}$-ball neighborhood of 0 and 1. □

Lemma 2 shows that $A'^{(11)}_{-22}$ is close to a projection, which enables us to apply the functional calculus of f to $A'^{(11)}_{-22}$.

Step 3. Apply the functional calculus of f to ${A'}_{-22}^{(11)}$. By Lemma 2, we have $\|f({A'}_{-22}^{(11)})-{A'}_{-22}^{(11)}\| < 3m^{1/3}\varepsilon^{1/3}$. Set ${A''}_{-22}^{(11)} = f({A'}_{-22}^{(11)})$, $A''_{-} = \begin{pmatrix} A''_{-11} & 0 \\ 0 & A''_{-22} \end{pmatrix}$, where $A''_{-11} = A'_{-11}$ and $A''_{-22} = \begin{pmatrix} {A''}_{-22}^{(11)} & 0 \\ 0 & {A''}_{+22}^{(22)} \end{pmatrix}$. By setting the diagonal entries of ${A'}_{+22}^{(11)}$ those are either smaller than 0 to be 0 and those are bigger than 1 to be 1, we obtain A''_{+}. From the construction of A''_{+} and A''_{-}, it is clear that $0 \le A''_{+}, A''_{-} \le 1$. Now it is time to estimate $\|(A''_{+} - {A''_{+}}^{2})(A''_{+} - A''_{-})\|$ and $\|(A''_{-} - {A''_{-}}^{2})(A''_{+} - A''_{-})\|$.

Theorem 3. $\|(A''_{+} - {A''_{+}}^{2})(A''_{+} - A''_{-})\| < 2\varepsilon^{1/2}$, $\|(A''_{-} - {A''_{-}}^{2})(A''_{+} - A''_{-})\| < 6m^{1/3}\varepsilon^{1/3}$.

Proof. Set

$$M''_{+} = A''_{+} - {A''_{+}}^{2} = \begin{pmatrix} 0 & 0 \\ 0 & M''_{+22} \end{pmatrix}, M''_{+22} = \begin{pmatrix} {M''}_{+22}^{(11)} & 0 \\ 0 & {M''}_{+22}^{(22)} \end{pmatrix},$$

where $\|M_{+22}^{(11)}\| \le \varepsilon^{1/2}$; and

$$N''_{+} = A''_{+} - A''_{-} = \begin{pmatrix} 0 & 0 \\ 0 & N''_{+22} \end{pmatrix}, N''_{+22} = \begin{pmatrix} {A''}_{+22}^{(11)} - {A''}_{-22}^{(11)} & 0 \\ 0 & 0 \end{pmatrix},$$

where $\|{A''}_{+22}^{(11)} - {A''}_{-22}^{(11)}\| \le 2$. Hence

$$\begin{aligned} &\|(A''_{+} - {A''_{+}}^{2})(A''_{+} - A''_{-})\| \\ =&\|M''_{+}N''_{+}\| = \|M''_{+22}N''_{+22}\| = \|{M''}_{+22}^{(11)}({A''}_{+22}^{(11)} - {A''}_{-22}^{(11)})\| \le 2\varepsilon^{1/2}. \end{aligned}$$

Let $P''_{+} = A''_{-} - {A''_{-}}^{2} = \begin{pmatrix} 0 & 0 \\ 0 & P''_{+22} \end{pmatrix}$, where $P''_{+22} = \begin{pmatrix} {P''}_{+22}^{(11)} & 0 \\ 0 & {P''}_{+22}^{(22)} \end{pmatrix}$. Since the spectrum of ${A'}_{-22}^{(11)}$ lies in the closed $3m^{1/3}\varepsilon^{1/3}$-ball neighborhood of 0 and 1, when ε is sufficiently small, we have $\|{P''}_{+22}^{(11)}\| < 3m^{1/3}\varepsilon^{1/3}$. Hence

$$\begin{aligned} &\|(A''_{-} - {A''_{-}}^{2})(A''_{+} - A''_{-})\| \\ =&\|P''_{+}N''_{+}\| = \|P''_{+22}N''_{+22}\| = \|{P''}_{+22}^{(11)}({A''}_{+22}^{(11)} - {A''}_{-22}^{(11)})\| < 6m^{1/3}\varepsilon^{1/3}. \end{aligned}$$

□

So far, we have got the parallel refinement $({A_+}'', {A_-}'')$ for the weakening idempotent pair (A_+, A_-).

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